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Long-Range Scalar Field in Conformally Flat Space–Time

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Consequences of a massless scalar field in conformally flat space–time are studied. Then a wide class of solutions of the scalar field is obtained.

1. INTRODUCTION

In a recent paper Penney (1976) presented a special class of solutions of the field equations for the coupled zero-mass scalar field and gravitational field in conformally flat space–time. The conformally flat space–time drastically reduces the number of unknown functions in Einstein's equations, but it turns out that in a region devoid of any matter content the space–time is essentially flat. However, such a situation gives conformal space–time a redeeming feature; one can think of a scalar gravity that acts only inside the source structures and may be considered an important agent in the formation of elementary structures (Das, 1971).

In this paper we propose to study the consequences of the conformal flatness subjected to Einstein's equations. Einstein's equations restrict the Riemann curvature tensor to a form that explicitly shows that outside the distribution the space–time is flat. A wide class of solutions of the field equations for the coupled zero-mass scalar field and gravitational field is then obtained; under a particular choice of some parameter this class of solutions reduces to Penney's solution.

2. BASIC EQUATIONS

We consider a space–time whose metric tensor is

$$g_{ij} = e^{\Psi}\eta_{ij} \tag{2.1}$$

where η_{ij} is the flat metric having signature $+2$ and where $\Psi = \Psi(x^i)$. Since equation (2.1) describes a conformally flat metric, we have

$$C^h{}_{ijk} = 0 \tag{2.2}$$

where the conformal curvature is

$$C_{hijk} = R_{hijk} + \tfrac{1}{2}(g_{h[j}R_{k]i} + g_{i[k}R_{j]h}) + (R/6)g_{h[k}g_{j]i} \tag{2.3}$$

where the square brackets denote antisymmetrization.

Einstein's equations are

$$R_{ij} - \tfrac{1}{2}g_{ij}R = -kT_{ij} \tag{2.4}$$

For a real zero rest-mass scalar field minimally coupled to the gravitational field, the energy momentum tensor is

$$T_{ij} = \phi_{,i}\phi_{,j} - \tfrac{1}{2}g_{ij}g^{kl}\phi_{,k}\phi_{,l} \tag{2.5}$$

From equations (2.2)–(2.5) one immediately gets

$$R_{hijk} = \tfrac{1}{2}k(g_{h[k}\phi_{,j]}\phi_{,i} + g_{i[j}\phi_{,k]}\phi_{,h} - \tfrac{2}{3}g^{kl}\phi_{,k}\phi_{,l}g_{h[j}g_{k]i}) \tag{2.6}$$

where equation (2.6) shows clearly that $R_{hijk} = 0$ in a region where $\phi = \text{const}$, which implies a flat space–time. Now the differential Bianchi identities satisfied by R_{hijk} yield

$$\phi_{,i}\phi_{,[j;k]} - \tfrac{1}{6}g^{ab}\phi_{,a}\phi_{,b;[k}g_{j]i} = 0 \tag{2.7}$$

where comma and semicolon denote ordinary derivative and coderivative. On contracting i and k, we get

$$g^{ij}\phi_{;ij} = 0 \tag{2.8}$$

The real scalar field of zero rest mass then satisfies the wave equation (2.8).

3. EXACT SOLUTIONS OF THE COUPLED EQUATIONS

The Ricci tensor for the metric (2.1) is computed (Synge, 1960) to be

$$R_{ij} = \Psi_{,ij} - \tfrac{1}{2}\Psi_{,i}\Psi_{,j} + \tfrac{1}{2}\eta_{ij}(\eta^{ab}\Psi_{,ab} + \eta^{ab}\Psi_{,a}\Psi_{,b}) \tag{3.1}$$

where $\Psi_{,i}$ is the partial derivative of Ψ. From equation (2.4), and using (2.5), we have

$$R_{ij} = -k\phi_{,i}\phi_{,j} \tag{3.2}$$

Taking the trace of equations (3.1) and (3.2) we get

$$\eta^{ij}(\Psi_{,ij} + \tfrac{1}{2}\Psi_{,i}\Psi_{,j}) = -(k/3)\eta^{ij}\phi_{,i}\phi_{,j} \tag{3.3}$$

A special class of solutions is studied by imposing a functional relationship of the form

$$\Psi = \Psi(\phi) \tag{3.4}$$

If we use equation (3.4), equation (3.3) leads to

$$\eta^{ij}\phi_{,i}\phi_{,j}[3\Psi'' - \tfrac{3}{2}\Psi'^2 + k] = 0 \tag{3.5}$$

where $k = 8\pi$ and primes denote derivative with respect to ϕ. Since $\eta^{ij}\phi_{,i}\phi_{,j} \neq 0$, $\Psi(\phi)$ satisfies the differential equation

$$3\Psi'' - \tfrac{3}{2}\Psi'^2 + k = 0 \tag{3.6}$$

The general solution of equation (3.6) is

$$\begin{aligned}\Psi = &-2 \ln \{[\sqrt{(k/6)} - \tfrac{1}{2}\eta] \exp [2\sqrt{(k/6)}(\phi - \xi)] + \sqrt{(k/6)} + \tfrac{1}{2}\eta\} \\ &+ 2\sqrt{(k/6)}(\phi - \xi) + M\end{aligned} \tag{3.7}$$

where ξ is a particular value of ϕ, and $\eta = (\Psi')_\xi$ is given by the boundary conditions, and M is the constant of integration that can be fixed by imposing the condition $\Psi = 0$ when $\phi = 0$. Then we can express equation (3.7) as

$$\Psi = -2 \ln \left\{\frac{C \exp [\sqrt{(2k/3)}(\phi - \xi)] + D}{C \exp [-\sqrt{(2k/3)}\xi] + D}\right\} + \sqrt{(2k/3)}\phi \tag{3.8}$$

where

$$C = \sqrt{(k/6)} - \tfrac{1}{2}\eta, \qquad D = \sqrt{(k/6)} + \tfrac{1}{2}\eta \tag{3.9}$$

Equation (3.8) shows the relation between Ψ and any ϕ that satisfies equation (2.8).

4. SOLUTIONS OF THE SCALAR FIELD

From equations (2.4), (2.5), (3.1), and (3.2) and using (3.6), one gets

$$\tfrac{2}{3}k\phi_{,i}\phi_{,j} + \Psi'\phi_{,ij} + \tfrac{1}{2}\eta_{ij}\eta^{ab}\Psi''\phi_{,a}\phi_{,b} = 0 \tag{4.1}$$

One particularly simple solution of equation (4.1) is obtained from (3.8) when $C = 0$. In this case

$$\Psi = \sqrt{(2k/3)}\phi \tag{4.2}$$

Substituting (4.2) in (4.1), one gets

$$\phi_{,ij} + \sqrt{(2k/3)}\phi_{,i}\phi_{,j} = 0 \tag{4.3}$$

The general solution of (4.3) has the form

$$\phi = \sqrt{(3/2k)} \ln (d_i x^i + 1) \tag{4.4}$$

where d_i are the constants of integration. In his paper Penney presented this wave solution of the system (2.4) and (2.5).

On taking

$$4\left[\frac{C}{D} \exp \left(-\sqrt{\frac{2k}{3}}\,\xi\right) + \frac{D}{C} \exp \left(\sqrt{\frac{2k}{3}}\,\xi\right) + 2\right]^{-1} = \pm a^2 \tag{4.5}$$

one can rewrite equation (3.8) as

$$\Psi' = \sqrt{\left(\frac{2k}{3}\right)}\phi - 2\ln\left\{\frac{1 \pm \sqrt{(1 \pm a^2)}}{2}\exp\left[\pm\sqrt{\left(\frac{2k}{3}\right)}\phi\right] \mp \frac{a^2}{2[1 \pm \sqrt{(1 \pm a^2)}]}\right\} \tag{4.6}$$

Substituting (4.6) in (4.1), one obtains the solution

$$\phi = -\sqrt{\frac{3}{2k}}\ln\left[\frac{C}{D}\exp\left(-\sqrt{\frac{2k}{3}}\,\xi\right) + 1\right] - \sqrt{\frac{3}{2k}}\ln\left[\frac{1}{1 \pm a^2F} - \frac{1}{2}\right] \tag{4.7}$$

where

$$F = \alpha\eta_{ij}(x^i + \xi^i)(x^j + \xi^j) \tag{4.8}$$

α and ξ^i are constants of integration.

Equation (4.6) furnish a general class of wave solutions of the system (2.4) and (2.5), of which (4.4) is a particular case.

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REFERENCES

Das, A. (1971). *J. Math. Phys.*, **12**, 232.
Penney, R. V. (1976). *Phys. Rev. D*, **14**, 910.
Synge, J. L. (1960). *Relativity: The General Theory*, North-Holland, Amsterdam.